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A new representation of Clifford algebras

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Abstract. A new method for the representation of Clifford algebras is presented, which does not make use of minimal one-sided ideals. It was developed by us as a generalisation of the work of Hestenes on the real Dirac-Clifford algebra of the γ matrices. Spinor spaces are subspaces isomorphic to a subalgebra of the original Clifford algebra. Inner products on spinor spaces are explicitly constructed and their isometries are studied.

1. Introduction

Clifford algebras are genuine geometric objects, since they are equivalent to exterior algebras with an inner product (Kähler 1960, 1961, 1962, Graf 1978). This is not so for their representations, better known as spinors. Although a Clifford bundle can be given on any differential manifold carrying an inner product, for the construction of the associated spinor bundle there are restrictions of a topological nature on the manifold (see Borel and Hirzebruch 1958, 1959, 1960, Geroch 1968). The differences in the geometric status between Clifford algebras and spinors can perhaps be better understood if one uses the methods of representation theory of abstract algebras (van der Waerden 1967), which are based on the concept of minimal left ideals. This was applied on Clifford algebras by Chevalley (1954). Note that, since minimal left ideals are defined by means of idempotents of the Clifford algebra, their use makes concrete calculations very complicated.

We present here a representation of Clifford algebras in themselves (Dimakis 1983, 1985), which do not use minimal left ideals. The representation spaces will be isomorphic linear subspaces of the original Clifford algebra, one of which will be additionally a Clifford subalgebra. Our method has some similarities to Cartan's representation theory of Lie algebras, since it is based on a maximal commutative subalgebra of the Clifford algebra. It was developed by us as a generalisation of Hestenes work (Hestenes 1966, 1967, 1973, 1985) on the real Dirac algebra of γ matrices.

We begin with the study of a real universal Clifford algebra $\mathscr{C}$ and obtain the results of Hestenes in case $\mathscr{C}$ becomes the real Dirac-Clifford algebra of γ matrices $\mathscr{D}$. Our arguments are different from those of Hestenes and can be generalised in a method applicable to any Clifford algebra. In order to obtain in this introduction as much information as is necessary to develop the general theory, we will also study the Majorana Clifford algebra $\mathscr{M}$.

Let $\mathscr{C}$ be generated by a fixed real finite-dimensional vector space X with a non-degenerated inner product g. We identify $\mathbb{R}$ and X with their images in $\mathscr{C}$ and write for the defining property of Clifford algebras, for $x, y \in X \subset \mathscr{C}$

$$xy + yx = 2g(x, y). \tag{1}$$

$\mathscr{C}$ is $\mathbb{Z}$-graded as a linear space and $\mathbb{Z}_2$-graded as an algebra. Let ω: $\mathscr{C} \to \mathscr{C}$ denote an involution of $\mathscr{C}$ defined for $x \in X \subset \mathscr{C}$ by

$$x^{\omega} := -x \tag{2a}$$

and for a, $b \in \mathscr{C}$ by

$$(ab)^{\omega} := a^{\omega} b^{\omega}. \tag{2b}$$

We call ω the *gradation involution* of $\mathscr{C}$ with respect to $X \subset \mathscr{C}$, since it defines a direct sum decomposition of $\mathscr{C}$:

$$\mathscr{C} = \mathscr{C}^{+} \oplus \mathscr{C}^{-} \tag{3}$$

with $\mathscr{C}^{\pm} := \{a \in \mathscr{C}\colon a^{\omega} = \pm a\}$ and $\mathscr{C}^{+}\mathscr{C}^{\pm} \subset \mathscr{C}^{\pm}$, $\mathscr{C}^{-}\mathscr{C}^{\pm} \subset \mathscr{C}^{\mp}$. Obviously $\mathscr{C}^{+}$ is a Clifford subalgebra of $\mathscr{C}$. We call $\mathscr{C}^{+}(\mathscr{C}^{-})$ the *even* (*odd*) component of $\mathscr{C}$ with respect to $X \subset \mathscr{C}$, since it consists of linear combinations of products of even (odd) numbers of elements of X.

Since $\mathscr{C}^{+}$ is closed under the Clifford product it is automatically a representation space of itself. We can extend it to a representation space of the whole $\mathscr{C}$, if we fix some odd element $u \in \mathscr{C}^{-}$ and introduce an operation of $\mathscr{C}$ on $\mathscr{C}^{+}$ defined for $a = a_{+} + a_{-} \in \mathscr{C}$, with $a_{+} \in \mathscr{C}^{+}$, $a_{-} \in \mathscr{C}^{-}$ and $\psi \in \mathscr{C}^{+}$, by

$$a \circ \psi := a_{+}\psi + a_{-}\psi u. \tag{4}$$

Since u is odd and the product of two odd elements is even, the right-hand side of (4) is even. In order for $\circ$ to define a representation of $\mathscr{C}$ it must satisfy

$$(ab) \circ \psi = a \circ (b \circ \psi) \tag{5}$$

for all elements a, $b \in \mathscr{C}$. In particular, if a, b are odd then ab is even and we conclude from (5)

$$u^2 = 1. \tag{6}$$

As we noted already, $\mathscr{C}^{+}$ is a Clifford subalgebra of $\mathscr{C}$ and thus a (universal) Clifford algebra for itself. Therefore we can apply the above procedure on it also. As we have done for $\mathscr{C}$ we take $\Omega\colon \mathscr{C}^{+} \to \mathscr{C}^{+}$ to be a gradation involution with respect to some generating subspace of $\mathscr{C}^{+}$. Again we decompose $\mathscr{C}^{+}$ into its even component $\mathscr{C}^{++}$ and its odd component $\mathscr{C}^{+-}$, with respect to Ω. We fix now an element $u' \in \mathscr{C}^{+-}$ with $u'^2 = 1$ and define for $A = A_{+} + A_{-} \in \mathscr{C}^{+}$ with $A_{+} \in \mathscr{C}^{++}$, $A_{-} \in \mathscr{C}^{+-}$ and $\xi \in \mathscr{C}^{++}$

$$A \circ \xi := A_{+}\xi + A_{-}\xi u'. \tag{7}$$

This again gives a representation of $\mathscr{C}^{+}$ on $\mathscr{C}^{++}$. The question arises now if we can extend this to become a representation of the original algebra $\mathscr{C}$. To do that we extend Ω to $\omega'\colon \mathscr{C} \to \mathscr{C}$ setting

$$\text{for } a \in \mathscr{C}^{+} \qquad a^{\omega'} := a^{\Omega} \tag{8a}$$

and

$$\text{for } a \in \mathscr{C}^{-} \qquad a^{\omega'} := (au)^{\Omega} u'. \tag{8b}$$

This defines an involution of $\mathscr{C}$ satisfying (i) u is even with respect to ω', that is

$$u^{\omega'} = u \tag{9}$$

and (ii) ω, ω' commute. We now decompose $\mathscr{C}^-$ also into its even component $\mathscr{C}^{-+}$ and its odd component $\mathscr{C}^{--}$ with respect to ω'. For $a = a_{++} + a_{+-} + a_{-+} + a_{--} \in \mathscr{C}$ and $\xi \in \mathscr{C}^{++}$ we define the operation of $\mathscr{C}$ on $\mathscr{C}^{++}$ by

$$a \circ \xi := a_{++}\xi + a_{+-}\xi u' + a_{-+}\xi u + a_{--}\xi uu'. \tag{10}$$

This must again satisfy (5) from which, for a, $b \in \mathscr{C}^{--}$, we find

$$uu' = u'u. \tag{11}$$

Note also that, since $u' \in \mathscr{C}^+$, this is even with respect to ω, that is

$$u'^{\omega} = u'. \tag{12}$$

The representation space $\mathscr{C}^{++}$ is a Clifford subalgebra of $\mathscr{C}^+$ and hence of $\mathscr{C}$. The other components in the direct sum decomposition

$$\mathscr{C} = \mathscr{C}^{++} \oplus \mathscr{C}^{+-} \oplus \mathscr{C}^{-+} \oplus \mathscr{C}^{--} \tag{13}$$

are also representation spaces of $\mathscr{C}$ under the circle operation. This is so because of

$$\mathscr{C}^{+-} = \mathscr{C}^{++}u' \qquad \mathscr{C}^{-+} = \mathscr{C}^{++}u \qquad \mathscr{C}^{--} = \mathscr{C}^{++}uu'$$

and the commutation property (11).

Application of the above procedure once more demands the finding of a new gradation involution ω'' of $\mathscr{C}$, which commutes with ω, ω' and satisfies

$$u^{\omega''} = u \qquad u'^{\omega''} = u'.$$

Further we must find an element u'', which is odd with respect to ω'', even with respect to ω and ω', commutes with u and u' and satisfies

$$u''^2 = 1.$$

Continuing in this way we expect this process to stop after a number of steps. This number will be an invariant of the particular Clifford algebra.

As a first example we take the real Majorana algebra $\mathscr{M}$ generated by $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\}$ with

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta'^{\mu\nu}$$

and

$$(\eta'^{\mu\nu}) := \mathrm{diag}(-1, +1, +1, +1).$$

If we set $u = \gamma^1$, then $u' = \gamma^0\gamma^2$ commutes with u and satisfies $u'^2 = 1$. We take ω (ω') to be the gradation involution of $\mathscr{M}$ with respect to the generating basis $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\}$ $\{\gamma^0, \gamma^0\gamma^1, \gamma^0\gamma^2, \gamma^0\gamma^3\}$). We have $\omega\omega' = \omega'\omega$ and

$$u^{\omega} = -u \qquad u^{\omega'} = u \qquad u'^{\omega} = u' \qquad u'^{\omega'} = -u'.$$

The even subalgebra of $\mathscr{M}$ with respect to ω and ω' is linearly generated by $\{1, \gamma^1\gamma^2, \gamma^1\gamma^3, \gamma^2\gamma^3\}$. There is no further element u'' and thus the representation process stops after the second step. To make concrete calculations we need a compact notation. Let l_2 denote a set with elements the ordered pairs $A = (A_1, A_2)$ with $A_1, A_2 = 0, 1$. For A, $B \in l_2$ we define addition and multiplication in l_2 by

$$A \dotplus B := (A_1 \dotplus B_1, A_2 \dotplus B_2) \tag{14a}$$

and

$$AB := (A_1B_1, A_2B_2) \tag{14b}$$

where $\dot{+}$ denotes addition modulo 2. We define also the length $|A|$ of $A \in l_2$ to be

$$|A| := A_1 + A_2 \in \mathbb{Z}. \tag{15}$$

It satisfies the relation

$$|A \dot{+} B| = |A| + |B| - 2|AB|. \tag{16}$$

We use now l_2 as an index set to define

$$\omega_A := \omega^{A_1} \omega'^{A_2} \tag{17}$$

where $\omega^0 := id$, $\omega^1 := \omega$ and composition of mappings is understood on the right-hand side of (17). Further we set

$$u_A := u^{A_1} u'^{A_2} \tag{18}$$

and introduce the idempotents

$$\pi_A := \tfrac{1}{4} \sum_{B \in l_2} (-1)^{|AB|} u_B. \tag{19}$$

$\mathscr{M}$ is decomposed in a direct sum

$$\mathscr{M} = \bigoplus_{A \in l_2} \mathscr{M}_A \tag{20}$$

where

$$\mathscr{M}_A := \{a \in \mathscr{M} : a^{\omega} = (-1)^{A_1} a,\ a^{\omega} = (-1)^{A_2} a\}. \tag{21}$$

The idempotents satisfy the relations

$$\pi_A \pi_B = \delta_{A,B} \pi_A \tag{22a}$$

and

$$\sum_{A \in l_2} \pi_A = 1. \tag{22b}$$

We also have

$$u_A \pi_B = (-1)^{|AB|} \pi_B. \tag{23}$$

Any element $a \in \mathscr{M}$ can be uniquely decomposed into

$$a = \sum_{A \in l_2} a_A \tag{24a}$$

with $a_A \in \mathscr{M}_A$ given by

$$a_A = \tfrac{1}{4} \sum_{B \in l_2} (-1)^{|AB|} a^{\omega_B}. \tag{24b}$$

From the definition of the circle operation we have for $\psi \in \mathscr{M}_{(0,0)}$

$$a \circ \psi = \sum_{A \in l_2} a_A \psi u_A$$

$$= \frac{1}{4} \sum_{A,B \in l_2} (-1)^{|AB|} a^{\omega_B} \psi u_A = \sum_{B \in l_2} a^{\omega_B} \psi \frac{1}{4} \sum_{A \in l_2} (-1)^{|AB|} u_A = \sum_{A \in l_2} a^{\omega_A} \psi \pi_A.$$

Thus we obtain

$$a \circ \psi = \sum_{A \in l_2} a^{\omega_A} \psi \pi_A. \tag{25}$$

This formula will be used later as definition for the circle operation.

Let $\mathscr{D}$ denote the real Dirac–Clifford algebra generated by the γ matrices $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\}$ with the defining relation

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$$

where

$$(\eta^{\mu\nu}) := \text{diag}(+1, -1, -1, -1).$$

We set

$$u := \gamma^0$$

and take ω to be the gradation involution of $\mathscr{D}$ with respect to the linear space generated by $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\}$. A linear basis of $\mathscr{D}^+$ is given by

$$\{1, \gamma^0\gamma^1, \gamma^0\gamma^2, \gamma^0\gamma^3, \gamma^1\gamma^2, \gamma^1\gamma^3, \gamma^2\gamma^3, \gamma^0\gamma^1\gamma^2\gamma^3\}.$$

The elements of $\mathscr{D}^+$ commuting with γ^0 are generated by $\{1, \gamma^1\gamma^2, \gamma^1\gamma^3, \gamma^2\gamma^3\}$. From them only 1 satisfies (6), which however is even with respect to all gradation involutions. Thus by $\mathscr{D}$ we stop after the first step. We have two representation spaces $\mathscr{D}^+$, $\mathscr{D}^-$, which have eight real dimensions. This is the number of the real components of a 4-spinor. The elements of $\mathscr{D}^+$ commuting with $u = \gamma^0$ constitute the basis of a subalgebra $\mathbb{H}'$ of $\mathscr{D}$, which is isomorphic to the skew field of quaternions. From the commutation property we obtain for $\lambda \in \mathbb{H}'$, $a \in \mathscr{D}$ and $\psi \in \mathscr{D}^+$

$$a \circ (\psi\lambda) = (a \circ \psi)\lambda. \tag{26}$$

This means we can interpret $\mathscr{D}^+$ as a right $\mathbb{H}'$-linear space. Taking this point of view $\mathscr{D}^+ \simeq \mathbb{H}^2$. Thus we obtain a 2×2 quaternionic representation of the γ matrices. In physics we use complex representations, therefore we pick an element, say $j \in \mathbb{H}'$, to represent the imaginary unit

$$j^2 = -1$$

and obtain an embedding of $\mathscr{D}$ in $\mathbb{C}^4$.

In the Majorana case no elements of $\mathscr{M}^{++} = \mathscr{M}_{(0,0)}$ other than 1 commute with both u and u'. Thus $\mathscr{M}_{(0,0)}$ is isomorphic to $\mathbb{R}^4$.

In what follows we formulate the above representation in a mathematically rigorous way applicable to all Clifford algebras. We restrict our analysis to real algebras, but the results apply also to complex ones. We begin with a presentation of real Clifford algebra theory, following mainly the book of Porteous (1969). A classification formula for real Clifford algebras will be derived, which will be of use in the next sections. According to this formula real Clifford algebras are completely characterised by three numbers. Next we introduce the circle operation and show under what conditions this gives a faithful and irreducible representation. This is the main part of the paper. It is divided into two parts, handling separately simple and not simple Clifford algebras. After the spinor spaces are obtained we construct inner products on them and study their isometry groups. Finally we apply the results obtained in some algebras of low dimension which are of interest in physics.

2. Classification of real Clifford algebras

$\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ denote the fields of real numbers, complex numbers and quaternions. If $\mathbb{K}$ denotes any of the above fields, then ${}^2\mathbb{K}$ will denote the ring $\mathbb{K} \times \mathbb{K}$ with addition and multiplication defined componentwise

$$(a, b)+(c, d):=(a+c, b+d)$$

$$(a, b)(c, d):=(ac, bd).$$

Following Porteous (1969) we call ${}^2\mathbb{K}$ a *double field.* If $\mathbb{B}$ denotes $\mathbb{K}$ or ${}^2\mathbb{K}$, then $\mathbb{B}(n)$ is the real algebra of $n \times n$ matrices with entries from $\mathbb{B}$. $\mathbb{R}^{p,q}$ will denote the orthogonal space $\mathbb{R}^{p+q}$ with inner product of signature (p, q), where p is the number of positive and q the number of negative signs.

There are many different but equivalent ways to define a Clifford algebra. We are not going to repeat here any of these definitions, or prove universality and existence. The reader can find in Chevalley (1954), Rasevskii (1957), Riesz (1958), Atiyah *et al* (1964), Hestenes (1966), Porteous (1969), Marcus (1975), Greub (1978) his favourite definition and proofs. We again use the conventions of Porteous and write $\mathbb{R}_{p,q}$ to denote the universal Clifford algebra for $\mathbb{R}^{p,q}$. One of the basic tools in the exposition of Porteous is the use of orthonormal subsets of Clifford algebras.

An *orthonormal subset* (ONS) of signature (p, q) of a real associative algebra A with unity 1 is a linearly free subset $Q=\{a_i \in A\colon i=1, \ldots, p+q\}$ of A, whose elements satisfy the relations

$$a_i a_j + a_j a_i = 0 \qquad \text{for } i \neq j \tag{27a}$$

$$a_i^2 = 1,\ a_j^2 = -1 \qquad \text{for } i=1, \ldots, p;\ j=p+1, \ldots, p+q. \tag{27b}$$

An *orthonormal basis* (ONB) of A is an ONS, which generates A.

The importance of ONS and ONB for Clifford algebras is based on the following fact.

Theorem 1. If A is a real associative algebra with unity 1 and has an ONS $Q=\{\vartheta^1, \ldots, \vartheta^{p+q}\}$ of signature (p, q) such that $\vartheta^1 \ldots \vartheta^{p+q} \neq \pm 1$, then $\mathbb{R}_{p,q}$ is isomorphic to the subalagebra of A generated by Q. If Q is an ONB of A then $A \simeq \mathbb{R}_{p,q}$.

For the classification of real Clifford algebras the following two lemmas are basic.

Lemma 1. If Q is an ONS of signature (i) $(p+1, q)$, (ii) $(p, q+1)$ of a real associative algebra with unity 1 and $a \in Q$ satisfies (i) $a^2=1$, (ii) $a^2=-1$, then the set

$$Q':=\{ba\colon b \in Q-\{a\}\} \cup \{a\}$$

is an ONS of signature (i) $(q+1, p)$, (ii) $(p, q+1)$.

(iii) If Q is an ONS of signature $(p, q+3)$ of a real associative algebra with unity 1 and $a_1, a_2, a_3 \in Q$ satisfy $(a_i)^2=-1$ for $i=1, 2, 3$, then the set

$$Q':=\{ba_1a_2a_3\colon b \in Q-\{a_1, a_2, a_3\}\} \cup \{a_1, a_2, a_3\}$$

is an ONS of signature $(q, p+3)$.

Lemma 2. Let A be a real associative algebra with unity 1, $\{e^1, e^2\}$ an ONS of A of signature (i) (1, 1), (ii) (2, 0) and (iii) (0, 2), and Q an ONS of A of signature (p, q), such that $\{e^1, e^2\} \cup Q$ is an ONS of A of signature (i) $(p+1, q+1)$, (ii) $(p+2, q)$ and (iii) $(p, q+2)$. Then from (i) $(e^1e^2)^2=1$, (ii) and (iii) $(e^1e^2)^2=-1$ and the fact that e^1e^2 commutes with elements of Q, we have $Q':=\{be^1e^2\colon b \in Q\}$ is an ONS of A of signature (i) (p, q), (ii) and (iii) (q, p) whose elements commute with e^1 and e^2. Conversely the existence of Q' implies the existence of Q.

From these two lemmas we obtain immediately

$$\mathbb{R}_{p+1,q} \simeq \mathbb{R}_{q+1,p} \tag{28a}$$

$$\mathbb{R}_{p,q+3} \simeq \mathbb{R}_{q,p+3} \tag{28b}$$

$$\mathbb{R}_{p+1,q+1} \simeq \mathbb{R}_{p,q} \otimes \mathbb{R}_{1,1} \tag{29a}$$

$$\mathbb{R}_{p+2,q} \simeq \mathbb{R}_{q,p} \otimes \mathbb{R}_{2,0} \tag{29b}$$

$$\mathbb{R}_{p,q+2} \simeq \mathbb{R}_{q,p} \otimes \mathbb{R}_{0,2}. \tag{29c}$$

From these and the relations

$$\mathbb{R}_{0,0} \simeq \mathbb{R} \qquad \mathbb{R}_{0,1} \simeq \mathbb{C} \qquad \mathbb{R}_{1,0} \simeq {}^2\mathbb{R} \qquad \mathbb{R}_{0,2} \simeq \mathbb{H} \qquad \mathbb{R}_{1,1} \simeq \mathbb{R}(2) \tag{30}$$

and

$$\mathbb{K} \otimes \mathbb{R}(p) \simeq \mathbb{K}(p) \qquad {}^2\mathbb{K} \otimes \mathbb{R}(p) \simeq {}^2\mathbb{K}(p) \qquad \mathbb{R}(p) \otimes \mathbb{R}(q) \simeq \mathbb{R}(pq) \tag{31}$$

$$\mathbb{C} \otimes \mathbb{C} \simeq {}^2\mathbb{C} \qquad \mathbb{C} \otimes \mathbb{H} \simeq \mathbb{C}(2) \qquad \mathbb{H} \otimes \mathbb{H} \simeq \mathbb{R}(4) \tag{32}$$

we can prove the following classification.

Theorem 2. For $p, q, m, k \in \mathbb{Z}$, $p, q \geq 0$, if (i) $p+q=2m$ and $p-q=8k$ or $8k+2$, then

$$\mathbb{R}_{p,q} \simeq \mathbb{R}_{m,m} \simeq \mathbb{R}(2^m).$$

(ii) $p+q=2m$ and $p-q=8k+4$ or $8k+6$, then

$$\mathbb{R}_{p,q} \simeq \mathbb{R}_{m-1,m+1} \simeq \mathbb{H}(2^{m-1}).$$

(iii) $p+q=2m+1$ and $p-q=8k+3$ or $8k+7$, then

$$\mathbb{R}_{p,q} \simeq \mathbb{R}_{m,m+1} \simeq \mathbb{C}(2^m).$$

(iv) $p+q=2m+1$ and $p-q=8k+1$, then

$$\mathbb{R}_{p,q} \simeq \mathbb{R}_{m+1,m} \simeq {}^2\mathbb{R}(2^m).$$

(v) $p+q=2m+1$ and $p-q=8k+5$, then

$$\mathbb{R}_{p,q} \simeq \mathbb{R}_{m-1,m+2} \simeq {}^2\mathbb{H}(2^{m-1}).$$

We can express the results of theorem 2 in a single formula with the aid of the following sequences.

For $n \in \mathbb{Z}$, let $n = 8k+m$, where $k, m \in \mathbb{Z}$ and $0 \leq m \leq 8$. We set

$$s(n) := \begin{cases} 1 & \text{for } m = 0, 4 \\ 0 & \text{otherwise.} \end{cases} \tag{33}$$

$$\varphi(n) := \begin{cases} 0 & \text{for } m = 0, 1, 2 \\ 1 & \text{for } m = 3, 7 \\ 2 & \text{for } m = 4, 5, 6. \end{cases} \tag{34}$$

$$\chi(n) := \begin{cases} 4k & \text{for } m = 0 \\ 4k+1 & \text{for } m = 1 \\ 4k+2 & \text{for } m = 2, 3 \\ 4k+3 & \text{for } m = 4, 5, 6, 7. \end{cases} \tag{35}$$

$\chi : \mathbb{Z} \to \mathbb{Z}$ is the extension of the *Radon–Hurwitz sequence* to negative integers.

For $p, q \geq 0$ we define the numbers

$$\zeta := \chi(p-q+2)+q-2 \qquad \eta := \varphi(p-q) \qquad \sigma := s(p-q-1) \qquad \kappa := 2^{\sigma} \tag{36}$$

and set $\mathbb{K}_\eta := \mathbb{R}, \mathbb{C}, \mathbb{H}$ for $\eta = 0, 1, 2$.

Theorem 3. For $p, q \geq 0$

$$\mathbb{R}_{p,q} \simeq \mathbb{R}_{\zeta,\zeta+\eta} \otimes \mathbb{R}_{\sigma,0} \simeq (\otimes^{\zeta}\mathbb{R}_{2,0}) \otimes \mathbb{R}_{0,\eta} \otimes \mathbb{R}_{\sigma,0} \simeq {}^{\kappa}\mathbb{K}_\eta(2^{\zeta}). \tag{37}$$

As becomes clear from the above formula $\sigma = 0$ or 1 shows if the algebra is simple or not, η characterises the field of numbers and ζ is the dimension of the real matrix algebra which is isomorphic to $\mathbb{R}_{p,q}$. As a byproduct of (37) we obtain from $\dim(\mathbb{R}_{p,q}) = \dim({}^2\mathbb{K}_\eta(2^{\zeta}))$, for $n \in \mathbb{Z}$:

$$\chi(n) = 1 + \tfrac{1}{2}(n - \varphi(n-2) - s(n-3))$$

and from this

$$\zeta = \tfrac{1}{2}(p+q-\eta-\sigma).$$

Before proceeding to the representation of Clifford algebras we need to improve our notation. For $n \in \mathbb{Z}$ we define the set

$$I_n := \{A := (A_1, \ldots, A_n) : A_i = 0, 1;\ i = 1, \ldots, n\}. \tag{38}$$

This becomes a commutative ring with unity with the operations of addition and multiplication defined by

$$A \dot{+} B := C \qquad \text{with} \qquad C_i = (A_i + B_i) \bmod 2;\ i = 1, \ldots, n$$

$$AB := C \qquad \text{with} \qquad C_i = A_i B_i;\ i = 1, \ldots, n.$$

The zero element of I_n is $0 := (0, \ldots, 0)$ and the unity $\Delta := (1, \ldots, 1)$. For $A \in I_n$ the length of $A = (A_1, \ldots, A_n)$ is defined by

$$|A| := A_1 + \ldots + A_n.$$

The length satisfies the relation

$$|A| + |B| = |A \dot{+} B| + 2|AB|. \tag{39}$$

I_n has 2^n elements and, as a ring, a structural similarity to the power set of $\{1, \ldots, n\}$. If $Q = \{e^1, \ldots, e^n\}$ is an ONB of $\mathbb{R}_{p,q}$ of signature (p, q), $p+q = n$, then we set

$$2^Q := \left\{ e_A := \prod_{i=1}^{n} (e^i)^{A_i} : A = (A_1, \ldots, A_n) \right\}. \tag{40}$$

2^Q has 2^n elements and is a linear basis of $\mathbb{R}_{p,q}$. (See Hagmark and Lounesto (1985) for history and further development of the use of I_n in Clifford algebras.)

The following identity:

$$\sum_{B \in I_n} (-1)^{|AB|} = 2^n \delta_{0,A} \tag{41}$$

will be of use in some calculations of the next section.

3. Representation of $\mathbb{R}_{p,q}$ with $\sigma = 0$

Q will denote an ONB of $\mathbb{R}_{p,q}$ of signature (p, q). ζ, η and σ are the numbers associated with $\mathbb{R}_{p,q}$ by (36). Here we restrict to the case $\sigma = 0$.

As is clear from lemmas 1, 2 and theorem 3 we can find an ONB $Q_0 \subset 2^Q$ of $\mathbb{R}_{p,q}$ of signature $(\zeta, \zeta+\eta)$, whose elements are homogeneous in 2^Q,

$$Q_0 := \{e_1, \ldots, e_\zeta; e_{\zeta+1}, \ldots, e_{\zeta+\eta}\} \tag{42}$$

with $e_i^2 = 1$ for $i = 1, \ldots, \zeta$ and $e_j^2 = -1$ for $j = \zeta+1, \ldots, \zeta+\eta$.

Theorem 4. The set

$$H := \{u_i := e_i e_{\zeta+i} : i = 1, \ldots, \zeta\} \tag{43}$$

has following properties:

(i) for all $u, u' \in H$ we have $u^2 = 1$ and $uu' = u'u$,

(ii) to every $u_i \in H$ there corresponds an ONB Q_i of $\mathbb{R}_{p,q}$, such that u_i is odd with respect to Q_i and even with respect to all Q_j with $j \neq i$.

Proof. (i) follows immediately from the definition of H. To prove (ii) we construct explicitly the ONB Q_i, $i = 1, \ldots, \zeta$. There are many different possibilities to do that; we parametrise one family of such constructions by $K = (K_1, \ldots, K_\zeta) \in l_\zeta$.

For $i = 1, \ldots, \zeta$, if $K_i = 0$ we set

$$Q_i := \{ae_{\zeta+i} : a \in Q_0 - \{e_{\zeta+i}\}\} \cup \{e_{\zeta+i}\} \tag{44a}$$

with signature $(\zeta, \zeta+\eta)$, and if $K_i = 1$ we set

$$Q_i := \{ae_i : a \in Q_0 - \{e_i\}\} \cup \{e_i\} \tag{44b}$$

with signature $(\zeta+\eta+1, \zeta-1)$. The case $\zeta = 0$ is trivial.

For all $i = 1, \ldots, \zeta$ it is obvious that $u_i \in Q_i$ and therefore is odd with respect to Q_i. Now for $j = 1, \ldots, \zeta$ with $j \neq i$, if $K_j = 0$, we have $u_i = e_i e_{\zeta+i} = (e_i e_{\zeta+j})(e_{\zeta+i} e_{\zeta+j})$, and if $K_j = 1$ then $u_i = -(e_i e_j)(e_{\zeta+i} e_j)$. In both cases u_i is even with respect to Q_j.

For $A \in l_\zeta$ we set

$$u_A := \prod_{i=1}^{\zeta} (u_i)^{A_i} \tag{45a}$$

$$\pi_A := \prod_{i=1}^{\zeta} \tfrac{1}{2}[1 + (-1)^{A_i} u_i]. \tag{45b}$$

These satisfy the relations

$$\pi_A \pi_B = \delta_{A,B} \pi_A \tag{46a}$$

$$\sum_{A \in l_\zeta} \pi_A = 1 \tag{46b}$$

and

$$u_A \pi_B = \pi_B u_A = (-1)^{|AB|} \pi_B. \tag{47}$$

The last relation leads with the aid of (41) to

$$u_A = \sum_{B \in l_\zeta} (-1)^{|AB|} \pi_B \tag{48a}$$

and

$$\pi_A = \frac{1}{2^\zeta} \sum_{B \in l_\zeta} (-1)^{|AB|} u_B. \tag{48b}$$

Lemma 3. If $a \in \mathbb{R}_{p,q}$ commutes with some $u_i \in H$, then $a = a_1 + a_2 u_i$, where a_1, a_2 are generated by $Q_0 - \{e_i, e_{\zeta+i}\}$.

Proof. We can write $a \in \mathbb{R}_{p,q}$ in the form

$$a = a_1 + be_i + ce_{\zeta+i} + a_2 e_i e_{\zeta+i},$$

where a_1, b, c, a_2 are elements of $\mathbb{R}_{p,q}$, which do not contain e_i, $e_{\zeta+i}$, and thus commute with u_i. Since u_i anticommutes with e_i and $e_{\zeta+i}$ we have

$$a = u_i a u_i = a_1 - be_i - ce_{\zeta+i} + a_2 e_i e_{\zeta+i}$$

and consequently $b = c = 0$.

Theorem 5. The set H defined in (43) is maximal in $\mathbb{R}_{p,q}$ with respect to its properties (i) and (ii) in theorem 4.

Proof. An element $a \in \mathbb{R}_{p,q}$ which commutes with all $u_i \in H$ can be written according to lemma 3 in the form

$$a = \sum_{A \in I_\zeta} a^A u_A$$

where a^A, $A \in I_\zeta$, are generated by $C := \{e_{2\zeta+1}, \ldots, e_{2\zeta+\eta}\}$.

For $\eta = 0$ we have $C = \emptyset$ and therefore $a^A = \lambda^A \in \mathbb{R}$ and a takes the form

$$a = \sum_{A \in I_\zeta} \lambda^A u_A \qquad \text{with} \qquad \lambda^A \in \mathbb{R}, A \in I_\zeta.$$

For $\eta = 1$ we have $C = \{e_{2\zeta+1}\}$ and therefore $a^A = \lambda_0^A + \lambda_1^A e_{2\zeta+1}$, with λ_0^A, $\lambda_1^A \in \mathbb{R}$ for all $A \in I_\zeta$. Thus $a = b_0 + b_1 e_{2\zeta+1}$, where $b_r = \Sigma_{A \in I_\zeta} \lambda_r^A u_A$ and from (48a) $b_r = \Sigma_{A \in I_\zeta} \mu_r^A \pi_A$ with $\mu_r^A \in \mathbb{R}$, $r = 0, 1$. Since a must have the properties given in theorem 4, we demand $a^2 = 1$, which leads to

$$a^2 = (b_0^2 - b_1^2) + (2b_0 b_1) e_{2\zeta+1} = 1.$$

From the linear independence of the summands we obtain

$$b_0^2 - b_1^2 = 1 \qquad b_0 b_1 = 0.$$

From the second of these equations and (46a) we find

$$b_0 b_1 = \sum_{A,B \in I_\zeta} \mu_0^A \mu_1^B \pi_A \pi_B = \sum_{A \in I_\zeta} \mu_0^A \mu_1^A \pi_A = 0.$$

Multiplying this equation with π_B for fixed $B \in I_\zeta$ we obtain

$$\mu_0^B \mu_1^B = 0 \qquad \text{for all } B \in I_\zeta.$$

Similarly from $b_0^2 - b_1^2 = 1$ we obtain

$$(\mu_0^B)^2 - (\mu_1^B)^2 = 1 \qquad \text{for all } B \in I_\zeta.$$

If now $\mu_1^B \neq 0$ for some $B \in I_\zeta$, then we must have $\mu_0^B = 0$ and finally $(\mu_1^B)^2 = -1$, which is impossible because the μ are real numbers. Thus $b_1 = 0$ and we obtain again

$$a = \sum_{A \in I_\zeta} \lambda^A u_A \qquad \text{with} \qquad \lambda^A \in \mathbb{R}, A \in I_\zeta.$$

A similar reasoning leads to the above form for $\eta = 2$.

Since a must be even with respect to all ONB Q_i, $i = 1, \ldots, \zeta$, we find $a = 1$, and hence there exists no ONB of $\mathbb{R}_{p,q}$ with respect to which a is odd. Thus H is maximal.

Relaxing property (ii) of theorem 4, we can prove that the set $H \cup \{e_{2\zeta+1}\}$ generates a maximal Abelian subalgebra of $\mathbb{R}_{p,q}$. This implies that, in a matrix representation of $\mathbb{R}_{p,q}$, the elements of this subalgebra are simultaneously diagonalisable. In this respect, and because of the special role played by H in the construction of the representation that follows, it is an analogue of the *Cartan subalgebra* in the representation theory of Lie algebras (Humphreys 1972).

We write ω_i, $i=1,\ldots,\zeta$, for the gradation involutions of $\mathbb{R}_{p,q}$ associated to Q_i, $i=1,\ldots,\zeta$. As can be proved easily on Q_0 these involutions commute and thus their compositions

$$\omega_A := \prod_{i=1}^{\zeta} (\omega_i)^{A_i} \tag{49}$$

are also involutions of $\mathbb{R}_{p,q}$. Here Π means composition of mappings and $(\omega_i)^0 := id$, $(\omega_i)^1 := \omega_i$. Combining these with the u and π we obtain

$$(u_A)^{\omega_B} = (-1)^{|AB|} u_A \tag{50a}$$

$$(\pi_A)^{\omega_B} = \pi_{A\dot{+}B}. \tag{50b}$$

We are now in the position to define the *circle operation* of $\mathbb{R}_{p,q}$ on itself as we did in (24) and (25) for the Majorana algebra. We set for a, $b \in \mathbb{R}_{p,q}$

$$a \circ b := \sum_{A \in l_\zeta} a^{\omega_A} b \pi_A. \tag{51}$$

Under this operation $\mathbb{R}_{p,q}$ becomes a left $\mathbb{R}_{p,q}$-module. Since distributivity is trivial we must check only

$$1 \circ b = b$$

which is a consequence of the definition and (46*b*) and for a, b, $c \in \mathbb{R}_{p,q}$

$$a \circ (b \circ c) = \sum_{A \in l_\zeta} a^{\omega_A} (b \circ c) \pi_A = \sum_{A,B \in l_\zeta} a^{\omega_A} b^{\omega_B} c \pi_B \pi_A = \sum_{A \in l_\zeta} (ab)^{\omega_A} c \pi_A = (ab) \circ c.$$

We look now for invariant submodules of $\mathbb{R}_{p,q}$. As is clear an involution induces a direct sum decomposition. We set for $A = (A_1, \ldots, A_\zeta) \in l_\zeta$

$$S_A := \{a \in \mathbb{R}_{p,q}: a^{\omega_i} = (-1)^{A_i} a\}. \tag{52}$$

Clearly

$$\mathbb{R}_{p,q} = \bigoplus_{A \in l_\zeta} S_A. \tag{53}$$

Lemma 4.

(i) $a \in S_A$ and $b \in S_B$ imply $ab \in S_{A\dot{+}B}$.

(ii) S_0 is a subalgebra of $\mathbb{R}_{p,q}$.

(iii) For all $A \in l_\zeta$, $u_A \in S_A$.

Theorem 6. S_0 is a universal Clifford algebra isomorphic to $\mathbb{R}_{\zeta-|K|,\eta+|K|}$, with $K \in l_\zeta$ defined in the proof of theorem 4.

Proof. From lemma 4 we have for $a \in S_0$, $au_A \in S_A$. Thus the linear spaces S_A, $A \in l_\zeta$, are isomorphic and from (53) we obtain

$$\dim S_0 = 2^{\zeta+\eta}.$$

From (44*a*, *b*) it is obvious that

$$Q_S := \{e_i : \text{with } i = K_j\zeta + j, j = 1, \ldots, \zeta\} \cup \{e_{2\zeta+1}, \ldots, e_{2\zeta+\eta}\} \tag{54}$$

is an ONB of S_0 of signature $(\zeta - |K|, \eta + |K|)$.

We set

$$s_i := e_{K_i\zeta+i} \qquad i = 1, \ldots, \zeta \qquad j_r := e_{2\zeta+r} \qquad r = 1, \ldots, \eta \tag{55}$$

then it is obvious that

$$u_i s_k = (-1)^{\delta_{ik}} s_k u_i \qquad u_i j_r = j_r u_i \tag{56}$$

for i, $k = 1, \ldots, \zeta$ and $r = 1, \ldots, \eta$.

Theorem 7. The set $\{a \in S_0 : a$ commutes with all $u_i \in H\}$ is a subalgebra of S_0 isomorphic to $\mathbb{K}_\eta$. We denote this subalgebra with $\mathbb{K}'_\eta$.

Proof. From lemma 3 we know that the elements of S_0, which commute with all $u_i \in H$, are generated by the ONS $\{j_r : r = 1, \ldots, \eta\}$ of signature $(0, \eta)$. Consequently this set is isomorphic to $\mathbb{R}_{0,\eta}$ for $\eta = 0, 1, 2$. As we know from (30) these Clifford algebras are isomorphic to the fields $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ respectively.

Theorem 8. The sets S_A, $A \in l_\zeta$, are left $\mathbb{R}_{p,q}$-modules under the circle operation and right $\mathbb{K}'_\eta$-linear spaces.

Proof. If $E_i \in l_\zeta$, $i = 1, \ldots, \zeta$, denote the standard basis of l_ζ, then we have $\omega_i = \omega_{E_i}$. From the definition of the circle operation (51) we find for a, $b \in \mathbb{R}_{p,q}$

$$(a \circ b)^{\omega_i} = \left(\sum_{A \in l_\zeta} a^{\omega_A} b \pi_A \right)^{\omega_{E_i}} = \sum_{A \in l_\zeta} a^{\omega_{A+E_i}} b^{\omega_{E_i}} \pi_{A \dot{+} E_i} = a \circ (b^{\omega_i}).$$

Thus if $b \in S_A$ and hence satisfies $b^{\omega_i} = (-1)^{A_i} b$, then $a \circ b$ satisfies the same conditions and hence is an element of S_A for all $a \in \mathbb{R}_{p,q}$. Therefore S_A, $A \in l_\zeta$, are invariant submodules of $\mathbb{R}_{p,q}$.

Since $\mathbb{K}'_\eta \subset S_0$ we have from lemma 4 for $b \in S_A$ and $\lambda \in \mathbb{K}'_\eta$ also $b\lambda \in S_A$. Furthermore the elements of $\mathbb{K}'_\eta$ commute with all u_i and hence with all π_A. Therefore

$$a \circ (b\lambda) = (a \circ b)\lambda. \tag{57}$$

In other words, we have proved that the sets S_A, $A \in l_\zeta$, are *representation* $\mathbb{R}_{p,q}$-modules over $\mathbb{K}'_\eta$ (see van der Waerden 1967).

For $A = (A_1, \ldots, A_\zeta) \in l_\zeta$ we set

$$s_A := \prod_{i=1}^{\zeta} (s_i)^{A_i} \tag{58}$$

and obtain the following identities:

$$u_A s_B = (-1)^{|AB|} s_B u_A \tag{59a}$$

$$s_A \pi_B = \pi_{A \dot{+} B} s_A \tag{59b}$$

$$u_A \circ a = u_A a u_A \qquad \text{for all } a \in \mathbb{R}_{p,q} \tag{59c}$$

$$\pi_A \circ s_B = \delta_{A,B} s_B \tag{59d}$$

$$s_A \circ a = s_A a \qquad \text{for all } a \in \mathbb{R}_{p,q}. \tag{59e}$$

Lemma 5.

(i) For $s \in S_0$ and $a \in \mathbb{R}$ we have $a \circ s = (as) \circ 1$.
(ii) For $a \in \mathbb{R}_{p,q}$, $a \circ 1 = 0$ implies $a\pi_0 = 0$.
(iii) For a, $b \in \mathbb{R}_{p,q}$, $(a \circ b)u_A = a \circ (bu_A)$.

Theorem 9. The representation $\mathbb{R}_{p,q}$-modules S_A, $A \in l_\zeta$, over $\mathbb{K}'_\eta$ are faithful.

Proof. Let $a \in \mathbb{R}_{p,q}$ be such that $a \circ s = 0$ for all $s \in S_0$. Then from lemma 5 we have $as\pi_0 = 0$. Setting $s = s_A$ for all $A \in l_\zeta$ and using (59*b*) we obtain $a\pi_A s_A = 0$ for all $A \in l_\zeta$. Since s_A, $A \in l_\zeta$, are invertible we have $a\pi_A = 0$ for all $A \in l_\zeta$, and from (46*b*) $a = 0$. Hence ker $S_0 = \{0\}$.

For $A \in l_\zeta$ the set $\{s_B u_A : B \in l_\zeta\} \cup (\mathbb{K}'_\eta u_A)$ is a linear basis of S_A. The above arguments and (47) lead again to ker $S_A = \{0\}$ for all $A \in l_\zeta$.

Theorem 10. The representation $\mathbb{R}_{p,q}$-modules S_A, $A \in l_\zeta$, over $\mathbb{K}'_\eta$ are simple.

Proof. Let $T \subset S_0$ be a linear subspace of S_0 over $\mathbb{K}'_\eta$. If T is invariant under the circle operation and $T \neq \{0\}$, then there exists $t \in T$, $t \neq 0$, such that for all $a \in \mathbb{R}_{p,q}$, $a \circ t \in T$. In particular for $a = u_i \in H$ $(a = s_i \in S_0)$, $i = 1, \ldots, \zeta$, $u_i \circ t = u_i t u_i$, $(s_i \circ t = s_i t)$ belongs to T.

We set $t_1 := t + u_1 t u_1$ if the right-hand side does not vanish and $t_1 := s_1 t$ otherwise. Thus we obtain $t_1 \neq 0$, $t_1 \in T$ such that t_1 commutes with u_1.

Applying this process ζ times we obtain an element $t_\zeta \neq 0$, $t_\zeta \in T$ such that t_ζ commutes with all elements of H. Since $T \subset S_0$ we obtain from theorem 7 that $t_\zeta \in \mathbb{K}'_\eta$ and hence it possesses an inverse $t_\zeta^{-1} \in S_0$. From $t_\zeta^{-1} \circ t_\zeta = 1$ we find that $1 \in T$ and therefore $T = S_0$.

For $A \neq 0$, let $T_A \subset S_A$ be invariant in S_A under the circle operation. Then from (iii) of lemmas 4 and 5 we have $T_A u_A$ is invariant in S_0. Thus $T_A u_A = S_0$ and therefore $T_A = S_A$.

We have shown that the sets S_A, $A \in l_\zeta$, defined in (52) are simple and faithful representation $\mathbb{R}_{p,q}$-modules over $\mathbb{K}_\eta$. We call these sets *spinor spaces* of $\mathbb{R}_{p,q}$. Since further S_0 is a subalgebra of $\mathbb{R}_{p,q}$, we call this the *spinor algebra.*

Before closing this section we show how a matrix representation can be obtained with the above method.

Every element $\psi \in S_0$ can be written in terms of the basis $\{s_A : A \in l_\zeta\}$ of S_0 in the form

$$\psi = \sum_{A \in l_\zeta} s_A \psi^A \qquad \psi^A \in \mathbb{K}'_\eta. \tag{60}$$

Setting

$$s^A := s_A^{-1} \tag{61}$$

we obtain from (59*d*) and $s_A^{-1} = \pm s_A$, $s_A s_B = \pm s_{A+B}$,

$$\psi^A = \pi_0 \circ (s^A \psi). \tag{62}$$

In this way we associate to every element ψ of S_0 a column vector $(\psi^A) \in (\mathbb{K}'_\eta)^{2^\zeta}$.

Similarly we associate to every element a of $\mathbb{R}_{p,q} a 2^\zeta \times 2^\zeta$ matrix over $\mathbb{K}'_\eta$ through

$$a \circ s_A = \sum_{B \in l_\zeta} s_B a^B_A \qquad a \to (a^B_A) \tag{63a}$$

where

$$a^B_A := (\pi_0 s^B a s_A) \circ 1. \tag{63b}$$

4. Representation of $\mathbb{R}_{p,q}$ with $\sigma=1$

The case $\sigma=1$ can occur only if $n=p+q$ is an odd number. Then if $Q=\{\vartheta^i : i=1,\ldots,n\}$ is an ONB of $\mathbb{R}_{p,q}$ of signature (p,q), ϑ_Δ, $\Delta:=(1,\ldots,1)\in l_n$ lies in the centre of $\mathbb{R}_{p,q}$. Furthermore $\sigma=1$ implies that $\vartheta_\Delta^2=1$ and $\mathbb{R}_{p,q}$ has two twosided ideals: $\mathbb{R}_{p,q}(1\pm\vartheta_\Delta)$. Thus $\mathbb{R}_{p,q}$ is not simple and therefore it cannot possess a faithful and irreducible representation (van der Waerden 1967). As becomes clear from theorem 2, we have here only the two possibilities $\eta=0$ and $\eta=2$. We study them separately.

4.1. $\eta=0$

From theorem 3 we have

$$\mathbb{R}_{p,q}\simeq\mathbb{R}_{\zeta,\zeta}\otimes\mathbb{R}_{1,0}\simeq\mathbb{R}_{\zeta,\zeta+1}.$$

We can construct therefore an ONB of signature $(\zeta,\ \zeta+1)$:

$$Q_0:=\{e_1,\ldots,e_{\zeta+1};\ e_{\zeta+2},\ldots,e_{2\zeta+1}\} \tag{64}$$

where $e_i^2=1$ and $e_j^2=-1$ for $i=1,\ldots,\zeta+1$ and $j=\zeta+2,\ldots,2\zeta+1$. As in (43) we define the set

$$H:=\{u_i=e_ie_{\zeta+i+1}: i=1,\ldots,\zeta\}. \tag{65}$$

This set has the properties of theorem 4, but it is not maximal with respect to these properties. The ONB associated to u_i, $i=1,\ldots,\zeta$, are constructed for some $K\in l_\zeta$ as in (44a, b). For $i=1,\ldots,\zeta$, if $K_i=0$ we set

$$Q_i:=\{ae_{\zeta+i+1}\colon a\in Q_0-\{e_{\zeta+i+1}\}\}\cup\{e_{\zeta+i+1}\} \tag{66a}$$

and, if $K_i=1$, we set

$$Q_i:=\{ae_i: a\in Q_0-\{e_i\}\}\cup\{e_i\}. \tag{66b}$$

In both cases the ONB Q_i, $i=1,\ldots,\zeta$, have the signature $(\zeta+1,\zeta)$.

As in the last section we define with the aid of the gradation involutions with respect to Q_i, $i=1,\ldots,\zeta$, the circle operation and construct the spinor spaces. For the spinor algebra S_0 we have the ONB

$$Q_S:=\{e_{K_i\zeta+i}: i=1,\ldots,\zeta\}\cup\{e_{\zeta+1}\} \tag{67}$$

of signature $(\zeta-|K|+1,\ |K|)$. We set

$$s_i:=e_{K_i\zeta+i} \qquad \text{for} \qquad i=1,\ldots,\zeta;\ \alpha:=e_{\zeta+1}. \tag{68}$$

We again have

$$s_iu_j=(-1)^{\delta_{ij}}u_js_i \qquad \alpha u_i=u_i\alpha$$

for $i,j=1,\ldots,\zeta$. The subset of S_0 whose elements commute with all $u_i\in H$ is generated by $\{\alpha\}$ and hence is isomorphic to ${}^2\mathbb{R}$, $\alpha\to(1,-1)$. We denote it by ${}^2\mathbb{R}'$. S_A, $A\in l_\zeta$, give again representation $\mathbb{R}_{p,q}$-modules over ${}^2\mathbb{R}'$, which are faithful but not simple, since H is not maximal. S_0 consists of two simple submodules: $S_0(1\pm\alpha)$.

Setting $u_0:=e_{\zeta+1}$, with associated ONB Q_0 and $H':=H\cup\{u_0\}$ we obtain a set maximal with respect to the properties of theorem 4. We can repeat the above construction taking as basis the set H'. This time we obtain an irreducible but not faithful representation.

4.2. $\eta = 2$

This case can be handled exactly like the preceding one. We give therefore only the definitions of objects needed to construct the representation.

From (37) we have

$$\mathbb{R}_{p,q} \simeq \mathbb{R}_{\zeta,\zeta+2} \otimes \mathbb{R}_{1,0} \simeq \mathbb{R}_{\zeta,\zeta+3}.$$

Let

$$Q_0 := \{e_1, \ldots, e_\zeta; e_{\zeta+1}, \ldots, e_{2\zeta+3}\} \tag{69}$$

be an ONB of $\mathbb{R}_{p,q}$ of signature $(\zeta, \zeta+3)$. We set again

$$H := \{u_i = e_i e_{\zeta+i} : i = 1, \ldots, \zeta\}. \tag{70}$$

The associated ONB are defined for some $k \in I_\zeta$ exactly as in (44*a*, *b*), where (44*a*) have signature $(\zeta, \zeta+3)$ and 44*b*) have signature $(\zeta+4, \zeta-1)$. Again the set H is not maximal. The spinor algebra has an ONB of signature $(\zeta - |K|, |K|+3)$

$$Q_S := \{s_i : i = 1, \ldots, \zeta\} \cup \{j_1, j_2, \alpha j_1 j_2\} \tag{71}$$

where

$$s_i := e_{K,\zeta+i} \qquad i = 1, \ldots, \zeta,$$
$$j_1 := e_{2\zeta+1} \qquad j_2 := e_{2\zeta+2}$$
$$\alpha := e_{2\zeta+1} e_{2\zeta+2} e_{2\zeta+3}. \tag{72}$$

The subset of S_0, whose elements commute with the elements of H is generated by $\{j_1, j_2, \alpha j_1 j_2\}$ and is isomorphic to ${}^2\mathbb{H}$. The sets S_A, $A \in I_\zeta$, are faithful but not simple representation $\mathbb{R}_{p,q}$-modules over ${}^2\mathbb{H}'$.

5. Inner products in the spinor algebra and their isometry groups

Q will denote an ONB of $\mathbb{R}_{p,q}$ of signature (p, q). With respect to Q we have the gradation involution ω and we define two anti-involutions ρ and σ as follows: for $a \in Q$, b, $c \in \mathbb{R}_{p,q}$

$$a^\rho := a \qquad (bc)^\rho := c^\rho b^\rho \tag{73a}$$

and

$$a^\sigma := -a \qquad (bc)^\sigma := c^\sigma b^\sigma. \tag{73b}$$

We call ρ *reversion* and σ *conjugation* of $\mathbb{R}_{p,q}$ with respect to Q. Obviously $\sigma = \rho\omega$. Every invertible element $x \in \mathbb{R}_{p,q}$ defines an involution through

$$a^x = x^{-1} a x \tag{74}$$

$a \in \mathbb{R}_{p,q}$. In particular if $x^2 = 1$, then $a^x = xax$.

In $\mathbb{R}_{0,1} \simeq \mathbb{C}$ conjugation coincides with complex conjugation $z \to \bar{z}$ and in $\mathbb{R}_{0,2} \simeq \mathbb{H}$ conjugation becomes quaternionic conjugation $q \to \bar{q}$ and reversion will be denoted by $q \to \tilde{q}$. In ${}^2\mathbb{K}$ we have also the *hyperbolic* involution defined by

$$hb : {}^2\mathbb{K} \to {}^2\mathbb{K} \qquad (a, b) \to (a, b)^{hb} := (b, a). \tag{75}$$

Let X be a right linear space over $\mathbb{B}$, where $\mathbb{B}=\mathbb{K}$ or ${}^2\mathbb{K}$ and $\mathbb{K}=\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, with $\chi:\mathbb{B}\to\mathbb{B}$ an anti-involution of $\mathbb{B}$. A mapping

$$X\times X\to\mathbb{B} \qquad (x,y)\to(x,y) \tag{76a}$$

is called a $\mathbb{B}^\chi$-*symmetric* ($\mathbb{B}^\chi$-*antisymmetric*) *inner product* of X if

$$(x,y+z)=(x,y)+(x,z) \qquad (x,y\lambda)=(x,y)\lambda \tag{76b}$$

and

$$(x,y)^\chi=(y,x) \quad ((x,y)^\chi=-(y,x)) \tag{76c}$$

for x, y, $z\in X$ and $\lambda\in\mathbb{B}$. From (76*b*, *c*) we obtain

$$(x\lambda,y)=\lambda^\chi(x,y). \tag{76d}$$

In the sequence we write $\bar{\mathbb{C}}$ for the field of complex numbers with conjugation, $\tilde{\mathbb{H}}$ for the field of quaternions with reversion and $\bar{\mathbb{H}}$ the same field with conjugation. $hb\mathbb{K}$ will denote the double field ${}^2\mathbb{K}$ with the hyperbolic involution (see Porteous (1969) for these definitions and notation).

Theorem 11. For some fixed $Y\in l_\zeta$ the mapping $S_0\times S_0\to\mathbb{B}$ defined by

$$(\psi,\varphi)\to(\psi,\varphi)_Y:=\pi_0\circ(s^Y\psi^\sigma\varphi) \tag{77}$$

where ψ, $\varphi\in S_0$ and $\mathbb{B}:={}^\kappa\mathbb{K}'_\eta$, is a *non-degenerate* $\mathbb{B}^{(\sigma s_Y)}$-symmetric or $\mathbb{B}^{(\sigma s_Y)}$-antisymmetric inner product on S_0 according to whether $s_Y^\sigma=s_Y$ or $s_Y^\sigma=-s_Y$.

Proof. Since s_Y are homogeneous in 2^Q we have $s_Y^\sigma=\varepsilon_Y s_Y$ with $\varepsilon_Y=\pm1$. We set also $s_Ys_Z=\varepsilon_{Y,Z}s_{Y\dotplus Z}$, $\varepsilon_{Y,Z}=\pm1$. Then we find from $(s_As_B)s_C=s_A(s_Bs_C)$ and $(s_As_B)^\sigma=\varepsilon_{A,B}s^\sigma_{A\dotplus B}$ two identities

$$\varepsilon_{A,B}\varepsilon_{A\dotplus B,C}=\varepsilon_{A,B\dotplus C}\varepsilon_{B,C} \qquad \varepsilon_{B,A}=\varepsilon_A\varepsilon_B\varepsilon_{A,B}\varepsilon_{A\dotplus B}.$$

In terms of the linear basis $\{s_A:A\in l_\zeta\}$ of S_0 we set

$$\psi=\sum_{A\in l_\zeta}s_A\psi^A \qquad \varphi=\sum_{A\in l_\zeta}s_A\varphi^A.$$

Substituting these expressions in (77) and using (59*d*) we obtain

$$(\psi,\varphi)_Y=\sum_{A\in l_\zeta}\varepsilon_A\varepsilon_{A,A\dotplus Y}(\psi^A)^{\sigma s_Y}\varphi^{A\dotplus Y}.$$

From this expression we obtain all properties of an inner product on S_0. In particular we find

$$(\psi,\varphi)_Y^{\sigma s_Y}=\varepsilon_Y(\varphi,\psi)_Y. \tag{78}$$

The inner product defined in (77) is *non-degenerate* if $(\psi,\varphi)_Y=0$ for all $\psi\in S_0$ implies $\varphi=0$. To prove that, we set $\psi=s_A$, $A\in l_\zeta$, in (77) and obtain

$$\pi_{A\dotplus Y}\circ\varphi=0 \qquad \text{for all } A\in l_\zeta.$$

Summing over $A\in l_\zeta$ and using (46*b*) we obtain $\varphi=0$.

Since $Y\in l_\zeta$ we obtain from (77) 2^ζ inner products on S_0. Not all of these inner products are independent (see Porteous 1969).

An element Λ of $\mathbb{R}_{p,q}$ will be called an *isometry* of the inner product $(\ ,\ )_Y$ if it satisfies

$$(\Lambda \circ \psi, \Lambda \circ \varphi)_Y = (\psi, \varphi)_Y \tag{79}$$

for all ψ, $\varphi \in S_0$.

Since $u_i \in H$ are homogeneous elements of 2^Q we have

$$u_i^\sigma = (-1)^{\Sigma_i} u_i \tag{80}$$

with $\Sigma_i = 0$ or 1 for $i = 1, \ldots, \zeta$. These numbers define an element $\Sigma := (\Sigma_1, \ldots, \Sigma_\zeta)$ of l_ζ. From (80) we obtain

$$u_A^\sigma = (-1)^{|A\Sigma|} u_A \qquad \pi_A^\sigma = \pi_{A+\Sigma}. \tag{81}$$

Theorem 12. $\Lambda \in \mathbb{R}_{p,q}$ is an isometry of $(,)_Y$ if and only if

$$\Lambda^{\sigma\omega_{Y+\Sigma}} \Lambda = 1. \tag{82}$$

Proof. Expanding the circle products in (79) and using the definition (77) of $(,)_Y$ we find after a lengthy calculation

$$[\pi_Y \psi^\sigma (\Lambda^{\sigma\omega_{Y+\Sigma}} \Lambda - 1)] \circ \varphi = 0$$

for all ψ, $\varphi \in S$. By an argument similar to that used to prove non-degeneracy of $(\ ,\ )_Y$ we obtain from this expression

$$(\Lambda^{\sigma\omega_{Y+\Sigma}} \Lambda - 1) \circ \varphi = 0$$

for all $\varphi \in S_0$. Since S_0 is faithful, we obtain immediately (82).

6. Application on some low-dimensional Clifford algebras

For small values of p, $q \geqslant 0$ we give now some examples of the new representation of $\mathbb{R}_{p,q}$.

6.1. *The Pauli algebra $\mathbb{R}_{3,0}$*

From (37) we find here

$$\mathbb{R}_{3,0} \simeq \mathbb{R}_{1,2} \simeq \mathbb{C}(2).$$

Let $Q = \{\sigma_1, \sigma_2, \sigma_3\}$ be an ONB of $\mathbb{R}_{3,0}$ of signature $(3, 0)$. Then

$$Q_0 = \{e_1 = \sigma_1 ; e_2 = \sigma_1\sigma_3, e_3 = \sigma_1\sigma_2\}$$

is an ONB of signature $(1, 2)$. The set H has here one element $u_1 = e_1 e_2 = \sigma_3$ with associated ONB $Q_1 = Q$ ($K = 0$). The spinor algebra is the even subalgebra of $\mathbb{R}_{3,0}$. An ONB for it of signature $(0, 2)$ is given by

$$Q_S = \{e_2 = \sigma_1\sigma_3, e_3 = \sigma_1\sigma_2\}.$$

Of the elements of Q_S only e_3 commutes with u_1. We set therefore

$$s_1 := e_2 = \sigma_1\sigma_3 \qquad j := e_3 = \sigma_1\sigma_2.$$

S_0 is isomorphic to $\mathbb{C}^2$ with linear basis over $\mathbb{C}$

$$s_0 = 1 \qquad s_1 = \sigma_1\sigma_3.$$

The circle operation of $\mathbb{R}_{3,0}$ on S_0 is defined in terms of the idempotents

$$\pi_0 = \tfrac{1}{2}(1+u_1) \qquad \pi_1 = \tfrac{1}{2}(1-u_1)$$

to be for $a \in \mathbb{R}_{3,0}$ and $\xi \in S_0$

$$a \circ \xi = a^{\omega_0}\xi\pi_0 + a^{\omega_1}\xi\pi_1 = a_+\xi + a_-\xi u_1$$

where ω_1 is the gradation involution with respect to Q_1, $\omega_0 = id$ and a_+ (a_-) is the even (odd) part of a with respect to ω_1. Setting

$$s_0 \to \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad s_1 \to \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad j \to i = \sqrt{-1}$$

we obtain with $i = 1, 2, 3$ and A, $B = 0, 1$ from

$$\sigma_i \circ s_A = s_B \sigma^B_{iA}$$

the Pauli matrices. A spinor $\xi \in S_0$ has the form

$$\xi = s_0\xi^0 + s_1\xi^1 = \alpha^0 + \beta^0\sigma_1\sigma_2 + \alpha^1\sigma_1\sigma_3 + \beta^1\sigma_2\sigma_3$$

where $\xi^A = \alpha^A + \beta^A j$, α^A, $\beta^A \in \mathbb{R}$, $A \in I_1 = \{0, 1\}$. For ξ, $\eta \in S_0$ we have two inner products:

$$(\xi, \eta)_0 = \pi_0 \circ (s^0 \xi^\sigma \eta) = \bar{\xi}^0\eta^0 + \bar{\xi}^1\eta^1$$

$$(\xi, \eta)_1 = \pi_1 \circ (s^1 \xi^\sigma \eta) = \xi^0\eta^1 - \xi^1\eta^0$$

of which the first is $\bar{\mathbb{C}}$-symmetric and the second $\mathbb{C}$-antisymmetric. From $u_1^\sigma = -u_1$ we find that for $\Lambda \in \mathbb{R}$ the mapping $\xi \to \Lambda \circ \xi$ is an isometry of $(\ ,\ )_0$ $((\ ,\ )_1)$ if and only if $\Lambda^\rho\Lambda = 1$ $(\Lambda^\sigma\Lambda = 1)$. The isometry groups are U(2) (SL(2; $\mathbb{C}$)).

6.2. The Majorana algebra $\mathbb{R}_{3,1}$

For this algebra we have $\zeta = 2$, $\eta = 0$, $\sigma = 0$ and therefore

$$\mathbb{R}_{3,1} \simeq \mathbb{R}_{2,2} \simeq \mathbb{R}(4).$$

Let $Q = \{\gamma^0, \gamma^1, \gamma^2, \gamma^3\}$ be an ONB of signature (3, 1) with $(\gamma^0)^2 = -1$, $(\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = 1$. An ONB of $\mathbb{R}_{3,1}$ of signature (2, 2) is given by

$$Q_0 = \{e_1 = \gamma^3, e_2 = \gamma^0\gamma^3; e_3 = \gamma^3\gamma^1, e_4 = \gamma^3\gamma^2\}.$$

From this we obtain

$$u_1 = e_1e_3 = \gamma^1 \qquad u_2 = e_2e_4 = \gamma^0\gamma^2$$

with associated ONB $Q_1 = Q$ and $Q_2 = \{\gamma^0, \gamma^0\gamma^3, \gamma^0\gamma^1, \gamma^0\gamma^2\}$ $(K = (0, 0))$. An ONB for the spinor algebra S_{00} is given by

$$Q_S = \{s_1 = \gamma^3\gamma^1, s_2 = \gamma^3\gamma^2\}.$$

No element of S_{00} other than the unit 1 commutes with both u_1, u_2. Therefore S_{00} is isomorphic to $\mathbb{R}^4$ with linear basis over $\mathbb{R}$

$$s_{00} = 1 \qquad s_{10} = \gamma^3\gamma^1 \qquad s_{01} = \gamma^3\gamma^2 \qquad s_{11} = \gamma^2\gamma^1.$$

The circle multiplication given in the introduction leads to a real representation of the γ matrices. For $\psi, \varphi \in S_{00}$ with $\psi = \Sigma_{A\in I_\zeta} s_A \psi^A$, $\varphi = \Sigma_{A\in I_\zeta} s_A \varphi^A$ we have four inner products:

$$(\psi, \varphi)_{00} = \psi^{00}\varphi^{00} + \psi^{10} + \varphi^{10} + \psi^{01}\varphi^{10} + \psi^{11}\varphi^{11}$$

$$(\psi, \varphi)_{10} = \psi^{00}\varphi^{10} - \psi^{10}\varphi^{00} - \psi^{01}\varphi^{11} + \psi^{11}\varphi^{01}$$

$$(\psi, \varphi)_{01} = \psi^{00}\varphi^{01} - \psi^{01}\varphi^{00} + \psi^{10}\varphi^{11} - \psi^{11}\varphi^{10}$$

$$(\psi, \varphi)_{11} = \psi^{00}\varphi^{11} - \psi^{10}\varphi^{01} + \psi^{01}\varphi^{10} - \psi^{11}\varphi^{00}.$$

$(\,,\,)_{00}$ is $\mathbb{R}$-symmetric and the remaining inner products are $\mathbb{R}$-antisymmetric. With respect to Q, $u_1^\sigma = -u_1$ and $u_2^\sigma = -u_2$ we have $\Sigma = (1, 1)$ and the mapping $\psi \to \Lambda \circ \psi$ is an isometry of

$$\begin{array}{ll} (\,,\,)_{00} & \text{if } \Lambda^{\sigma\omega_{11}}\Lambda = 1 \\ (\,,\,)_{10} & \text{if } \Lambda^{\sigma\omega_{01}}\Lambda = 1 \\ (\,,\,)_{01} & \text{if } \Lambda^{\sigma\omega_{10}}\Lambda = \Lambda^{\rho}\Lambda = 1 \\ (\,,\,)_{11} & \text{if } \Lambda^{\sigma\omega_{00}}\Lambda = \Lambda^{\sigma}\Lambda = 1. \end{array}$$

The isometry groups are $O(4; \mathbb{R})$ for $(\,,\,)_{00}$ and $Sp(4, \mathbb{R})$ for all others.

6.3. *The Dirac algebra* $\mathbb{R}_{1,3}$

For this signature we have $\zeta = 1$, $\eta = 2$ and $\sigma = 0$. Consequently

$$\mathbb{R}_{1,3} \simeq \mathbb{H}(2).$$

Let $Q = \{\gamma^0; \gamma^1, \gamma^2, \gamma^3\}$ be an ONB of signature $(1, 3)$, where $(\gamma^0)^2 = 1$ and $(\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1$. We set

$$Q_0 = \{\gamma^3\gamma^0; \gamma^3\gamma^1, \gamma^3\gamma^2, \gamma^3\}$$

and obtain $u_1 = \gamma^0$ with associated ONB $Q_1 = Q$. An ONB for the spinor algebra is given by $Q_S = \{\gamma^3\gamma^0; \gamma^3\gamma^1, \gamma^3\gamma^2\}$. The subalgebra of S_0, whose elements commute with u_1, is linearly generated by $\{1, j_1 = \gamma^2\gamma^3, j_2 = \gamma^3\gamma^1, j_3 = \gamma^1\gamma^2\}$ and is isomorphic to $\mathbb{H}$. S_0 is therefore isomorphic to $\mathbb{H}^2$ and has $\{s_0 = 1, s_1 = \gamma^3\gamma^0\}$ as linear basis over $\mathbb{H}'$. Using the circle operation and setting

$$s_0 \to \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad s_1 \to \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad j_1 \to i_1 \qquad j_2 \to i_2 \qquad j_3 \to i_3$$

we obtain a quaternionic 2×2 matrix representation of the Dirac algebra:

$$\gamma^0 \to \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \gamma^1 \to \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} i_2 \qquad \gamma^2 \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} i_1 \qquad \gamma^3 \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

For $\psi = s_0\psi^0 + s_1\psi^1$, $\varphi = s_0\varphi^0 + s_1\varphi^1 \in S_0$ we have two inner products:

$$(\psi, \varphi)_0 = \bar{\psi}^0\varphi^0 - \bar{\psi}^1\varphi^1 \qquad \bar{\mathbb{H}}\text{-symmetric}$$

$$(\psi, \varphi)_1 = \tilde{\psi}^0\varphi^1 - \tilde{\psi}^1\varphi^0 \qquad \tilde{\mathbb{H}}\text{-antisymmetric}$$

which are equivalent (see Porteous 1969). Their isometries are determined by $\Lambda^\rho\Lambda = 1$ and $\Lambda^\sigma\Lambda = 1$, both giving $Sp(1, 1; \mathbb{H})$ as an isometry group.

6.4. The de Sitter-Clifford algebra $\mathbb{R}_{1,4}$

Here we have $\zeta = 1$, $\eta = 2$ and $\sigma = 1$. Hence

$$\mathbb{R}_{1,4} \simeq \mathbb{R}_{1,3} \otimes \mathbb{R}_{1,0} \simeq {}^2\mathbb{H}(2).$$

Let $Q = \{e_1; e_2, e_3, e_4, e_5\}$ be an ONB of $\mathbb{R}_{1,4}$ of signature $(1, 4)$. We set

$$Q_0 = \{e_2e_1; e_2, e_2e_3, e_2e_4, e_2e_5\}$$

and obtain $u_1 = e_1$ with associated ONB $Q_1 = Q$. An ONB for the spinor algebra S_0 is given by

$$Q_S = \{e_2e_1, e_2e_3, e_2e_4, e_2e_5\}$$

where the last three elements of it commute with u_1. We set therefore, in accordance to § 4.2,

$$s_0 = 1 \qquad s_1 = e_1e_2 \qquad j_1 = e_2e_3$$
$$j_2 = e_2e_4 \qquad j_3 = e_3e_4 \qquad \alpha = e_2e_3e_4e_5.$$

The set $\{j_1, j_2; \alpha\}$ generates the double field ${}^2\mathbb{H}'$. With the aid of the circle operation and setting

$$1 \to (1, 1) \qquad \alpha \to (1, -1)$$

we obtain the matrix representation:

$$e_1 \to \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}(1, 1) \qquad e_2 \to \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}(1, 1) \qquad e_3 \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}(i_1, i_1)$$
$$e_4 \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}(i_2, i_2) \qquad e_5 \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}(-i_3, i_3).$$

We have two inner products. For ψ, $\varphi \in S_0$ we obtain from (77)

$$(\psi, \varphi)_0 = \bar{\psi}^0\varphi^0 - \bar{\psi}^1\varphi^1 \qquad {}^2\bar{\mathbb{H}}\text{-symmetric}$$
$$(\psi, \varphi)_1 = (\tilde{\psi}^0)^{hb}\varphi^1 - (\tilde{\psi}^1)^{hb}\varphi^0 \qquad hb\tilde{\mathbb{H}}\text{-antisymmetric.}$$

$(\ ,\ )_0$ is reducible (cf Porteous 1969). Since $u_1^\sigma = -u_1$ we find that $\psi \to \Lambda \circ \psi$ is an isometry of $(\ ,\ )_0$ ($(\ ,\ )_1$) if $\Lambda^\rho\Lambda = 1$ ($\Lambda^\sigma\Lambda = 1$). The isometry groups are respectively $\mathrm{Sp}(1, 1; \mathbb{H}) \otimes \mathrm{Sp}(1, 1; \mathbb{H})$ and $\mathrm{GL}(2; \mathbb{H})$.

The representation constructed above is faithful but not irreducible. We obtain from it an irreducible but not faithful representation if we multiply the matrices given above with $(1, 0)$ or $(0, 1)$.

6.5. The anti-de Sitter-Clifford algebras $\mathbb{R}_{2,3}$ and $\mathbb{R}_{3,2}$

(i) $\mathbb{R}_{2,3}$. For this algebra we find $\zeta = 2$, $\eta = 1$, $\sigma = 0$ and

$$\mathbb{R}_{2,3} \simeq \mathbb{C}(4).$$

Take $Q = \{e_1, e_2; e_3, e_4, e_5\}$ to be an ONB of $\mathbb{R}_{2,3}$ of signature $(2, 3)$ and set

$$Q_0 = \{e_4e_1, e_4e_2; e_4, e_4e_3, e_4e_5\}.$$

We then have $u_1 = e_1$ and $u_2 = e_2e_3$. The associated ONB are $Q_1 = Q$ and $Q_2 = \{e_1e_3, e_2e_3; e_3, e_4e_3, e_5e_5\}$ $(K = (0, 0))$. The spinor algebra has the ONB

$$Q_S = \{e_4e_1, e_4e_2; e_4e_5\}$$

of signature (2, 1). Element e_4e_5 of Q_S commutes with both u, we set therefore $s_1 = e_4e_2$, $s_2 = e_4e_1$ and $j = e_4e_5$. This leads to the definition

$$s_{00} = 1 \qquad s_{10} = e_4e_2 \qquad s_{01} = e_4e_1 \qquad s_{11} = e_2e_1$$

which gives a linear basis of S_0 over $\mathbb{C}'$ generated by $\{j\}$. We have four inner products:

$$(\psi, \varphi)_{00} = \bar{\psi}^{00}\varphi^{00} - \bar{\psi}^{10}\varphi^{10} - \bar{\psi}^{01}\varphi^{01} + \bar{\psi}^{11}\varphi^{11} \qquad \bar{\mathbb{C}}\text{-symmetric}$$

$$(\psi, \varphi)_{10} = \psi^{00}\varphi^{10} - \psi^{10}\varphi^{00} + \psi^{01}\varphi^{11} - \psi^{11}\varphi^{01} \qquad \mathbb{C}\text{-antisymmetric}$$

$$(\psi, \varphi)_{01} = \psi^{00}\varphi^{01} - \psi^{10}\varphi^{11} - \psi^{01}\varphi^{00} + \psi^{11}\varphi^{10} \qquad \mathbb{C}\text{-antisymmetric}$$

$$(\psi, \varphi)_{11} = \bar{\psi}^{00}\varphi^{11} - \bar{\psi}^{10}\varphi^{01} + \bar{\psi}^{01}\varphi^{10} - \bar{\psi}^{11}\varphi^{00} \qquad \bar{\mathbb{C}}\text{-antisymmetric.}$$

Since $\Sigma = (1, 1)$ we obtain for the isometries $\psi \to \Lambda \circ \psi$ $\Lambda^{\rho\omega_{01}}\Lambda = 1$ for $(\ ,\ )_{00}$, $\Lambda^{\sigma\omega_{01}}\Lambda = 1$ for $(\ ,\ )_{10}$, $\Lambda^{\rho}\Lambda = 1$ for $(\ ,\ )_{01}$, $\Lambda^{\sigma}\Lambda = 1$ for $(\ ,\ )_{11}$. The isometry groups are U(2, 2) for $(\ ,\ )_{00}$ and $(\ ,\ )_{11}$, and Sp(4; $\mathbb{C}$) for $(\ ,\ )_{10}$ and $(\ ,\ )_{01}$. Finally we note that $\mathbb{R}_{4,1} \simeq \mathbb{R}_{2,3}$.

(ii) $\mathbb{R}_{3,2}$. For this algebra we have $\zeta = 2$, $\eta = 0$, $\sigma = 1$ and

$$\mathbb{R}_{3,2} \simeq {}^2\mathbb{R}(2).$$

Let $Q = \{e_1, e_2, e_3; e_4, e_5\}$ be an ONB of $\mathbb{R}_{3,2}$ of signature (3, 2). We set

$$Q_0 = \{e_4e_1, e_4e_2, e_4e_3; e_4, e_4e_5\}$$

and find $u_1 = e_1$, $u_2 = e_2e_5$. The associated ONB are $Q_1 = Q$ and $Q_2 = \{e_1e_5, e_2e_5, e_3e_5; e_5, e_4e_5\}$ $(K = (0, 0))$. The spinor algebra has the ONB

$$Q_S = \{e_4e_1, e_4e_2, e_4e_3\}$$

of signature (3, 0). We set further $s_1 = e_4e_1$, $s_2 = e_4e_2$ and $\alpha = e_4e_3$. This leads to

$$s_{00} = 1 \qquad s_{10} = e_4e_1 \qquad s_{01} = e_4e_2 \qquad s_{11} = e_1e_2$$

which is a linear basis of S_0 over ${}^2\mathbb{R}'$ generated by $\{\alpha\}$. We have four inner products:

$$(\psi, \varphi)_{00} = (\psi^{00})^{hb}\varphi^{00} - (\psi^{10})^{hb}\varphi^{10} - (\psi^{01})^{hb}\varphi^{10} + (\psi^{11})^{hb}\varphi^{11}$$

$$(\psi, \varphi)_{10} = \psi^{00}\varphi^{10} - \psi^{10}\varphi^{00} + \psi^{01}\varphi^{11} - \psi^{11}\varphi^{01}$$

$$(\psi, \varphi)_{01} = \psi^{00}\varphi^{01} - \psi^{10}\varphi^{11} - \psi^{01}\varphi^{00} + \psi^{11}\varphi^{10}$$

$$(\psi, \varphi)_{11} = (\psi^{00})^{hb}\varphi^{11} - (\psi^{10})^{hb}\varphi^{01} + (\psi^{01})^{hb}\varphi^{10} - (\psi^{11})^{hb}\varphi^{00}$$

which are $hb\mathbb{R}$-symmetric, ${}^2\mathbb{R}$-antisymmetric, ${}^2\mathbb{R}$-antisymmetric and $hb\mathbb{R}$-antisymmetric, respectively. $(\ ,\ )_{10}$ and $(\ ,\ )_{01}$ are equivalent and reducible inner products, and their isometry group is Sp(4; $\mathbb{R}$)$\otimes$Sp(4; $\mathbb{R}$). Similarly $(\ ,\ )_{00}$ and $(\ ,\ )_{11}$ are equivalent with isometry group GL(4; $\mathbb{R}$) (see Porteous 1969).

6.6. *The conformal Clifford algebra* $\mathbb{R}_{2,4}$

We have $\zeta = 2$, $\eta = 2$, $\sigma = 0$ and

$$\mathbb{R}_{2,4} \simeq \mathbb{H}(4).$$

Let $Q = \{e_1, e_2; e_3, e_4, e_5, e_6\}$ be an ONB of signature (2, 4). We set

$$Q_0 = \{e_4e_1, e_4e_2; e_4, e_4e_3, e_4e_5, e_4e_6\}.$$

From this ONB we find $u_1 = e_1$ and $u_2 = e_2 e_3$ with associated ONB $Q_1 = Q$ and $Q_2 = \{e_1e_3, e_2e_3; e_3, e_4e_3, e_5e_3, e_6e_3\}$. For the spinor algebra we have

$$Q_S = \{e_1e_4, e_2e_4; e_4e_5, e_4e_6\}.$$

The elements of Q_S which commute with the u are e_4e_5 and e_4e_6. We set $s_1 = e_2e_4$, $s_2 = e_1e_4$ and $j_1 = e_4e_5$, $j_2 = e_4e_6$ and

$$s_{00} = 1 \qquad s_{10} = s_1 \qquad s_{01} = s_2 \qquad s_{11} = e_2e_1 \qquad j_3 = e_5e_6.$$

$\{s_A : A \in I_2\}$ is a linear basis of S over $\mathbb{H}'$, which is generated by $\{j_1, j_2\}$. We have four inner products:

$$(\psi, \varphi)_{00} = \bar{\psi}^{00}\varphi^{00} - \bar{\psi}^{10}\varphi^{10} - \bar{\psi}^{01}\varphi^{01} + \bar{\psi}^{11}\varphi^{11} \qquad \bar{\mathbb{H}}\text{-symmetric}$$

$$(\psi, \varphi)_{10} = \tilde{\psi}^{00}\varphi^{10} - \tilde{\psi}^{10}\varphi^{00} + \tilde{\psi}^{01}\varphi^{11} - \tilde{\psi}^{11}\varphi^{01} \qquad \tilde{\mathbb{H}}\text{-antisymmetric}$$

$$(\psi, \varphi)_{01} = \tilde{\psi}^{00}\varphi^{01} - \tilde{\psi}^{10}\varphi^{11} - \tilde{\psi}^{01}\varphi^{00} + \tilde{\psi}^{11}\varphi^{10} \qquad \tilde{\mathbb{H}}\text{-antisymmetric}$$

$$(\psi, \varphi)_{11} = \bar{\psi}^{00}\varphi^{11} - \bar{\psi}^{10}\varphi^{01} + \bar{\psi}^{01}\varphi^{10} - \bar{\psi}^{11}\varphi^{00} \qquad \bar{\mathbb{H}}\text{-antisymmetric.}$$

The first three inner products are equivalent and have Sp(2, 2; $\mathbb{H}$) as isometry group. $(\ ,\ )_{11}$ has the isometry group O(4; $\mathbb{H}$) with $\Lambda^\sigma\Lambda = 1$.

In all examples discussed above the inner products $(\ ,\ )_1$ or $(\ ,\ )_{11}$ have been constructed to correspond to the conjugation anti-involution. The results found are in agreement with those of Porteous (1969).

7. Conclusions

We introduced a new representation of Clifford algebras in a constructive way. Given a Clifford algebra $\mathbb{R}_{p,q}$ one needs only find an ONB Q_0 of signature $(\zeta, \zeta+\eta)$ for $\sigma = 0$, $(\zeta+1, \zeta)$ for $\sigma = 1$, $\eta = 0$ or $(\zeta, \zeta+3)$ for $\sigma = 1$ and $\eta = 2$ to begin the construction. Lemma 1 gives a constructive way to obtain an ONB of $\mathbb{R}_{p,q}$ of any signature (r, s), for which $\mathbb{R}_{r,s} \simeq \mathbb{R}_{p,q}$ holds. After the determination of Q_0 the method of § 3 gives a canonical way to construct the circle operation and the spinor algebra. This does not mean that the representation obtained thus depends only on Q_0. As becomes obvious from (51) and (52) the representation depends primarily on H and the family of gradation involutions ω_i, $i = 1, \ldots, \zeta$ introduced in (49) and associated to the ONB constructed in theorem 4. It is not easy to control how the representation operation σ and the spinor algebra S_0 changes under changes of H and ω_i. However, from theorem 3 we know that all representations obtained by such changes are equivalent†, being equivalent to ${}^\kappa\mathbb{K}_\eta(2^\zeta)$. An answer to the above question can be given perhaps by working in a higher level of abstraction, freeing the method from its constructive character. Although a characterisation of the subalgebra of $\mathbb{R}_{p,q}$ generated by H independent of Q_0 is almost obvious from the properties in (i) of theorem 4 it is not yet clear to the author how to characterise the subset H of it and the associated family of gradation involutions independent of Q_0.

† This is not in contradiction to the well known fact that not simple Clifford algebras ($\sigma = 1$) have two inequivalent representations. The use made here of the double fields ${}^2\mathbb{K}$ allows us to handle both representations in a unified way as a single object.

In physics Clifford algebras arise mainly as structures associated to a given linear space with an inner product (e.g. the tangent space of a manifold carrying a metric). The question arises then of how the representation obtained here changes under diffeomorphisms and local isometries. We will treat this question and related problems in a separate paper.

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